convenient means of calculating the force on a body (or the energy for the body in the field)—for it is the *force* (and energy) and not the *potential* that is the physically meaningful quantity. Thus, in some problems, it may be easier to calculate the force directly, rather than computing a potential and then taking the gradient. The advantage of using the potential method is that the potential is a *scalar* quantity*: We need not deal with the added complication of sorting out the components of a vector until the gradient operation is performed. In direct calculations of the force, the components must be carried through the entire computation. Some skill, then, is necessary in choosing the particular approach to use. For example, if a problem has a particular symmetry that, from physical considerations, allows us to determine that the force has a certain direction, then the choice of that direction as one of the coordinate directions reduces the vector calculation to a simple scalar calculation. In such a case, the direct calculation of the force may be sufficiently straightforward to obviate the necessity of using the potential method. Every problem requiring a force must be examined to discover the easiest method of computation.

EXAMPLE 5.4

Consider a thin uniform disk of mass M and radius a. Find the force on a mass m located along the axis of the disk.

Solution. We solve this problem by using both the potential and direct force approaches. Consider Figure 5.9. The differential potential $d\Phi$ at a distance z is

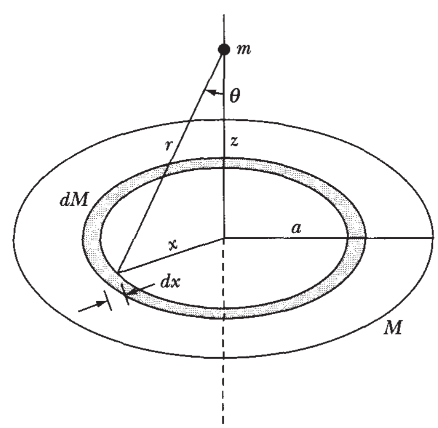

FIGURE 5-9 Example 5.4. We use the geometry shown here to find the gravitational force on a point mass m due to a thin uniform disk of mass M.

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*We shall see in Chapter 7 another example of a scalar function from which vector results may be obtained. This is the **Lagrangian function,** which, to emphasize the similarity, is sometimes (mostly in older treatments) called the *kinetic potential.*

given by

$$d\Phi = -G\frac{dM}{r} \tag{5.41}$$

The differential mass dM is a thin ring of width dx, because we have azimuthal symmetry.

$$dM = \rho dA = \rho 2\pi x\, dx \tag{5.42}$$

$$\begin{aligned} d\Phi &= -2\pi\rho G\frac{x\,dx}{r} = -2\pi\rho G\frac{x\,dx}{(x^2+z^2)^{1/2}} \\ \Phi(z) &= -\pi\rho G\int_0^a \frac{2x\,dx}{(x^2+z^2)^{1/2}} = -2\pi\rho G(x^2+z^2)^{1/2}\Big|_0^a \\ &= -2\pi\rho G[(a^2+z^2)^{1/2} - z] \end{aligned} \tag{5.43}$$

We find the force from

$$\mathbf{F} = -\nabla U = -m\nabla\Phi \tag{5.44}$$

From symmetry, we have only a force in the z direction,

$$F_z = -m\frac{\partial\Phi(z)}{\partial z} = +2\pi m\rho G\left[\frac{z}{(a^2+z^2)^{1/2}} - 1\right] \tag{5.45}$$

In our second method, we compute the force directly using Equation 4.2:

$$d\mathbf{F} = -Gm\frac{dM'}{r^2}\mathbf{e}_r \tag{5.46}$$

where dM' refers to the mass of a small differential area more like a square than a thin ring. The vectors complicate matters. How can symmetry help? For every small dM' on one side of the thin ring of width dx, another dM' exists on the other side that exactly cancels the horizontal component of $d\mathbf{F}$ on m. Similarly, all horizontal components cancel, and we need only consider the vertical component of $d\mathbf{F}$ along z.

$$dF_z = \cos\theta\,|d\mathbf{F}| = -mG\frac{\cos\theta\, dM'}{r^2}$$

and, because $\cos\theta = z/r$,

$$dF_z = -mG\frac{z\,dM'}{r^3}$$

Now we integrate over the mass $dM' = \rho 2\pi x\,dx$ around the ring and obtain

$$dF_z = -mG\rho\frac{2\pi xz\,dx}{r^3}$$

and

$$\begin{aligned} F_z &= -\pi m\rho G z \int_0^a \frac{2x\,dx}{(z^2 + x^2)^{3/2}} \\ &= -\pi m\rho G z \left[\frac{-2}{(z^2 + x^2)^{1/2}} \right] \Bigg|_0^a \\ &= 2\pi m\rho G \left[\frac{z}{(a^2 + z^2)^{1/2}} - 1 \right] \end{aligned} \tag{5.47}$$

which is identical to Equation 5.45. Notice that the value of F_z is negative, indicating that the force is downward in Figure 5-9 and attractive.

5.5 Ocean Tides

The ocean tides have long been of interest to humans. Galileo tried unsuccessfully to explain ocean tides but could not account for the timing of the approximately two high tides each day. Newton finally gave an adequate explanation. The tides are caused by the gravitational attraction of the ocean to both the Moon and the Sun, but there are several complicating factors.

The calculation is complicated by the fact that the surface of Earth is not an inertial system. Earth and Moon rotate about their center of mass (and move about the Sun), so we may regard the water nearest the Moon as being pulled away from Earth, and Earth as being pulled away from the water farthest from the Moon. However, Earth rotates while the Moon rotates about Earth. Let's first consider only the effect of the Moon, adding the effect of the Sun later. We will assume a simple model whereby Earth's surface is completely covered with water, and we shall add the effect of Earth's rotation at an appropriate time. We set up an inertial frame of reference $x'y'z'$ as shown in Figure 5.10a. We let M_m be the mass of the Moon, r the radius of a circular Earth, and D the distance from the center of the Moon to the center of Earth. We consider the effect of both the Moon's and Earth's gravitational attraction on a small mass m placed on the surface of Earth. As displayed in Figure 5-10a, the position vector of the mass m from the Moon is $\mathbf{R}$, from the center of Earth is $\mathbf{r}$, and from our inertial system $\mathbf{r}'_m$. The position vector from the inertial system to the center of Earth is $\mathbf{r}'_E$. As measured from the inertial system, the force on m, due to the earth and the Moon, is

$$m\ddot{\mathbf{r}}'_m = -\frac{GmM_E}{r^2}\mathbf{e}_r - \frac{GmM_m}{R^2}\mathbf{e}_R \tag{5.48}$$

Similarly, the force on the center of mass of Earth caused by the Moon is

$$M_E\ddot{\mathbf{r}}'_E = -\frac{GM_EM_m}{D^2}\mathbf{e}_D \tag{5.49}$$

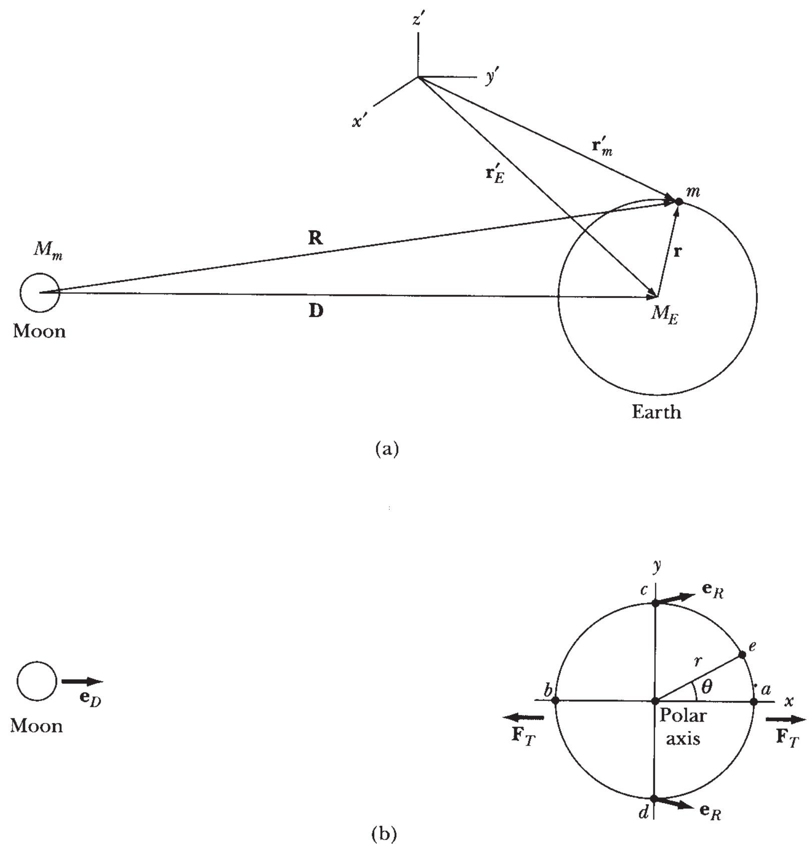

FIGURE 5-10 (a) Geometry to find ocean tides on Earth due to the Moon. (b) Polar view with the polar axis along the z-axis.

We want to find the acceleration $\ddot{r}$ as measured in the noninertial system placed at the center of Earth. Therefore, we want

$$\begin{aligned}
\ddot{r} = \ddot{r}'_m - \ddot{r}'_E &= \frac{m\ddot{r}'_m}{m} - \frac{M_E \ddot{r}'_E}{M_E} \\
&= -\frac{GM_E}{r^2}\mathbf{e}_r - \frac{GM_m}{R^2}\mathbf{e}_R + \frac{GM_m}{D^2}\mathbf{e}_D \\
&= -\frac{GM_E}{r^2}\mathbf{e}_r - GM_m\left(\frac{\mathbf{e}_R}{R^2} - \frac{\mathbf{e}_D}{D^2}\right) \qquad \textbf{(5.50)}
\end{aligned}$$

The first part is due to Earth, and the second part is the acceleration from the **tidal** force, which is responsible for producing the ocean tides. It is due to the difference between the Moon's gravitational pull at the center of Earth and on Earth's surface.

We next find the effect of the tidal force at various points on Earth as noted in Figure 5-10b. We show a polar view of Earth with the polar axis along the z-axis. The tidal force $\mathbf{F}_T$ on the mass m on Earth's surface is

$$\mathbf{F}_T = -GmM_m\left(\frac{\mathbf{e}_R}{R^2} - \frac{\mathbf{e}_D}{D^2}\right) \tag{5.51}$$

where we have used only the second part of Equation 5.50. We look first at point a, the farthest point on Earth from the Moon. Both unit vectors $\mathbf{e}_R$ and $\mathbf{e}_D$ are pointing in the same direction away from the Moon along the x-axis. Because $R > D$, the second term in Equation 5.51 predominates, and the tidal force is along the $+x$-axis as shown in Figure 5-10b. For point b, $R < D$ and the tidal force has approximately the same magnitude as at point a because $r/D \ll 1$, but is along the $-x$-axis. The magnitude of the tidal force along the x-axis, F_{Tx}, is

$$F_{Tx} = -GmM_m\left(\frac{1}{R^2} - \frac{1}{D^2}\right) = \mp GmM_m\left(\frac{1}{(D+r)^2} - \frac{1}{D^2}\right)$$

$$= -\frac{GmM_m}{D^2}\left(\frac{1}{\left(1+\frac{r}{D}\right)^2} - 1\right)$$

We expand the first term in brackets using the $(1 + x)^{-2}$ expansion in Equation D.9.

$$F_{Tx} = -\frac{GmM_m}{D^2}\left[1 - 2\frac{r}{D} + 3\left(\frac{r}{D}\right)^2 - \cdots - 1\right] = +\frac{2GmM_m r}{D^3} \tag{5.52}$$

where we have kept only the largest nonzero term in the expansion, because $r/D = 0.02$.

For point c, the unit vector $\mathbf{e}_R$ (Figure 5-10b) is not quite exactly along $\mathbf{e}_D$, but the x-axis components approximately cancel, because $R \simeq D$ and the x-components of $\mathbf{e}_R$ and $\mathbf{e}_D$ are similar. There will be a small component of $\mathbf{e}_R$ along the y-axis. We approximate the y-component of $\mathbf{e}_R$ by $(r/D)\mathbf{j}$, and the tidal force at point c, call it $\mathbf{F}_{Ty}$, is along the y-axis and has the magnitude

$$F_{Ty} = -GmM_m\left(\frac{1}{D^2}\frac{r}{D}\right) = -\frac{GmM_m r}{D^3} \tag{5.53}$$

Note that this force is along the $-y$-axis toward the center of Earth at point c. We find similarly at point D the same magnitude, but the component of $\mathbf{e}_R$ will be along the $-y$-axis, so the force itself, with the sign of Equation 5.53, will be along the $+y$-axis toward the center of Earth. We indicate the tidal forces at points a, b, c, and d on Figure 5-11a.

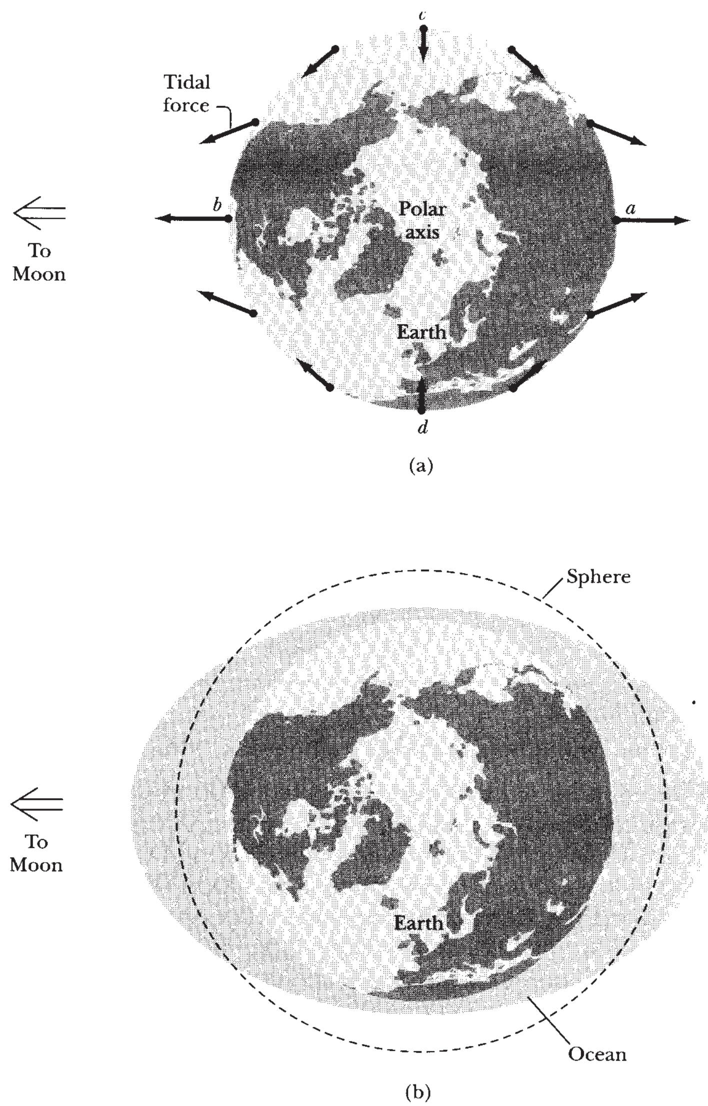

FIGURE 5-11 (a) The tidal forces are shown at various places on Earth's surface including the points *a*, *b*, *c*, and *d* of Figure 5-10. (b) An exaggerated view of Earth's ocean tides.

We determine the force at an arbitrary point *e* by noting that the *x*- and *y*-components of the tidal force can be found by substituting x and y for r in F_{Tx} and F_{Ty}, respectively, in Equations 5.52 and 5.53.

$$F_{Tx} = \frac{2GmM_m x}{D^3}$$

$$F_{Ty} = -\frac{GmM_m y}{D^3}$$

Then at an arbitrary point such as e, we let $x = r\cos\theta$ and $y = r\sin\theta$, so we have

$$F_{Tx} = \frac{2GmM_m r\cos\theta}{D^3} \tag{5.54a}$$

$$F_{Ty} = -\frac{GmM_m r\sin\theta}{D^3} \tag{5.54b}$$

Equations 5.54a and b give the tidal force around Earth for all angles θ. Note that they give the correct result at points a, b, c, and d.

Figure 5-11a gives a representation of the tidal forces. For our simple model, these forces lead to the water along the y-axis being more shallow than along the x-axis. We show an exaggerated result in Figure 5-11b. As Earth makes a revolution about its own axis every 24 hours, we will observe two high tides a day.

A quick calculation shows that the Sun's gravitational attraction is about 175 times stronger than the Moon's on Earth's surface, so we would expect tidal forces from the Sun as well. The tidal force calculation is similar to the one we have just performed for the Moon. The result (Problem 5-18) is that the tidal force due to the Sun is 0.46 that of the Moon, a sizable effect. Despite the stronger attraction due to the Sun, the gravitational force gradient over the surface of Earth is much smaller, because of the much larger distance to the Sun.

EXAMPLE 5.5

Calculate the maximum height change in the ocean tides caused by the Moon.

Solution. We continue to use our simple model of the ocean surrounding Earth. Newton proposed a solution to this calculation by imagining that two wells be dug, one along the direction of high tide (our x-axis) and one along the direction of low tide (our y-axis). If the tidal height change we want to determine is h, then the difference in potential energy of mass m due to the height difference is mgh. Let's calculate the difference in work if we move the mass m from point c in Figure 5-12 to the center of Earth and then to point a. This work W done by gravity must equal the potential energy change mgh. The work W is

$$W = \int_{r+\delta_1}^{0} F_{Ty}\,dy + \int_{0}^{r+\delta_2} F_{Tx}\,dx$$

where we use the tidal forces F_{Ty} and F_{Tx} of Equations 5.54. The small distances δ_1 and δ_2 are to account for the small variations from a spherical Earth, but these values are so small they can be henceforth neglected. The value for W becomes

$$W = \frac{GmM_m}{D^3}\left[\int_{r}^{0}(-y)dy + \int_{0}^{r} 2x\,dx\right]$$
$$= \frac{GmM_m}{D^3}\left(\frac{r^2}{2} + r^2\right) = \frac{3GmM_m r^2}{2D^3}$$

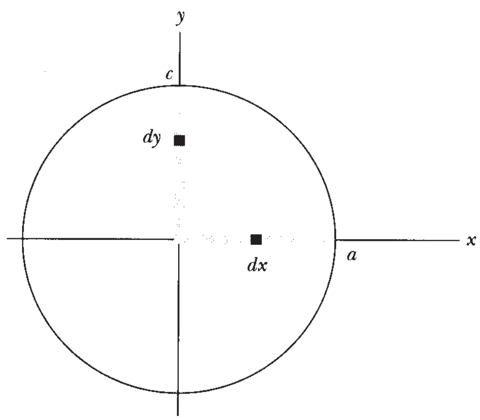

FIGURE 5-12 Example 5.5. We calculate the work done to move a point mass m from point c to the center of Earth and then to point a.

Because this work is equal to mgh, we have

$$mgh = \frac{3GmM_m r^2}{2D^3}$$

$$h = \frac{3GM_m r^2}{2gD^3} \quad \textbf{(5.55)}$$

Note that the mass m cancels, and the value of h does not depend on m. Nor does it depend on the substance, so to the extent Earth is plastic, similar tidal effects should be (and are) observed for the surface land. If we insert the known values of the constants into Equation 5.55, we find

$$h = \frac{3(6.67 \times 10^{-11}\,\mathrm{m^3/kg \cdot s^2})(7.350 \times 10^{22}\,\mathrm{kg})(6.37 \times 10^6\,\mathrm{m})^2}{2(9.80\,\mathrm{m/s^2})(3.84 \times 10^8\,\mathrm{m})^3} = 0.54\ \mathrm{m}$$

The highest tides (called *spring* tides) occur when Earth, the Moon, and the Sun are lined up (new moon and full moon), and the smallest tides (called *neap* tides) occur for the first and third quarters of the Moon when the Sun and Moon are at right angles to each other, partially cancelling their effects. The maximum tide, which occurs every 2 weeks, should be $1.46h = 0.83$ m for the spring tides.

An observer who has spent much time near the ocean has noticed that typical oceanshore tides are greater than those calculated in Example 5.5. Several other effects come into play. Earth is not covered completely with water, and the continents play a significant role, especially the shelfs and narrow estuaries. Local effects can be dramatic, leading to tidal changes of several meters. The tides in midocean, however, are similar to what we have calculated. Resonances can affect the natural oscillation of the bodies of water and cause tidal changes.

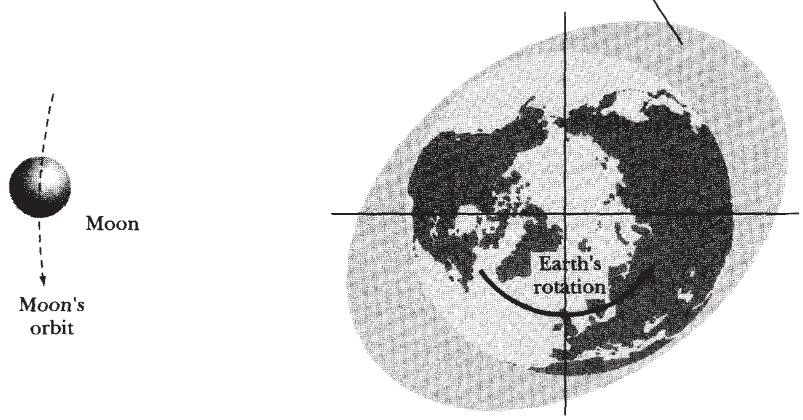

FIGURE 5-13 Some effects cause the high tides to not be exactly along the Earth-Moon axis.

Tidal friction between water and Earth leads to a significant amount of energy loss on Earth. Earth is not rigid, and it is also distorted by tidal forces.

In addition to the effects just discussed, remember that as Earth rotates, the Moon is also orbiting Earth. This leads to the result that there are not quite exactly two high tides per day, because they occur once every 12 h and 26 min (Problem 5-19). The plane of the moon's orbit about Earth is also not perpendicular to Earth's rotation axis. This causes one high tide each day to be slightly higher than the other. The tidal friction between water and land mentioned previously also results in Earth "dragging" the ocean with it as Earth rotates. This causes the high tides to be not quite along the Earth-Moon axis, but rather several degrees apart as shown in Figure 5-13.

PROBLEMS

5-1. Sketch the equipotential surfaces and the lines of force for two point masses separated by a certain distance. Next, consider one of the masses to have a fictitious negative mass $-M$. Sketch the equipotential surfaces and lines of force for this case. To what kind of physical situation does this set of equipotentials and field lines apply? (Note that the lines of force have *direction*; indicate this with appropriate arrows.)

5-2. If the field vector is independent of the radial distance within a sphere, find the function describing the density $\rho = \rho(r)$ of the sphere.

5-3. Assuming that air resistance is unimportant, calculate the minimum velocity a particle must have at the surface of Earth to escape from Earth's gravitational field. Obtain a numerical value for the result. (This velocity is called the *escape velocity*.)

5-4. A particle at rest is attracted toward a center of force according to the relation $F = -mk^2/x^3$. Show that the time required for the particle to reach the force center from a distance d is d^2/k.

5-5. A particle falls to Earth starting from rest at a great height (many times Earth's radius). Neglect air resistance and show that the particle requires approximately $\frac{9}{11}$ of the total time of fall to traverse the first half of the distance.

5-6. Compute directly the gravitational force on a unit mass at a point exterior to a homogeneous sphere of matter.

5-7. Calculate the gravitational potential due to a thin rod of length l and mass M at a distance R from the center of the rod and in a direction perpendicular to the rod.

5-8. Calculate the gravitational field vector due to a homogeneous cylinder at exterior points on the axis of the cylinder. Perform the calculation **(a)** by computing the force directly and **(b)** by computing the potential first.

5-9. Calculate the potential due to a thin circular ring of radius a and mass M for points lying in the plane of the ring and exterior to it. The result can be expressed as an elliptic integral.* Assume that the distance from the center of the ring to the field point is large compared with the radius of the ring. Expand the expression for the potential and find the first correction term.

5-10. Find the potential at off-axis points due to a thin circular ring of radius a and mass M. Let R be the distance from the center of the ring to the field point, and let θ be the angle between the line connecting the center of the ring with the field point and the axis of the ring. Assume $R \gg a$ so that terms of order $(a/R)^3$ and higher may be neglected.

5-11. Consider a massive body of arbitrary shape and a spherical surface that is exterior to and does not contain the body. Show that the average value of the potential due to the body taken over the spherical surface is equal to the value of the potential at the center of the sphere.

5-12. In the previous problem, let the massive body be inside the spherical surface. Now show that the average value of the potential over the surface of the sphere is equal to the value of the potential that would exist on the surface of the sphere if all the mass of the body were concentrated at the center of the sphere.

5-13. A planet of density ρ_1 (spherical core, radius R_1) with a thick spherical cloud of dust (density ρ_2, radius R_2) is discovered. What is the force on a particle of mass m placed within the dust cloud?

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*See Appendix B for a list of some elliptic integrals.

5-14. Show that the gravitational self-energy (energy of assembly piecewise from infinity) of a uniform sphere of mass M and radius R is

$$U = -\frac{3}{5}\frac{GM^2}{R}$$

5-15. A particle is dropped into a hole drilled straight through the center of Earth. Neglecting rotational effects, show that the particle's motion is simple harmonic if you assume Earth has uniform density. Show that the period of the oscillation is about 84 min.

5-16. A uniformly solid sphere of mass M and radius R is fixed a distance h above a thin infinite sheet of mass density ρ_s (mass/area). With what force does the sphere attract the sheet?

5-17. Newton's model of the tidal height, using the two water wells dug to the center of Earth, used the fact that the pressure at the bottom of the two wells should be the same. Assume water is incompressible and find the tidal height difference h, Equation 5.55, due to the Moon using this model. (*Hint*: $\int_0^{y_{max}} \rho g_y \, dy = \int_0^{x_{max}} \rho g_x \, dx$; $h = x_{max} - y_{max}$, where $x_{max} + y_{max} = 2R_{earth}$, and R_{earth} is Earth's median radius.)

5-18. Show that the ratio of maximum tidal heights due to the Moon and Sun is given by

$$\frac{M_m}{M_s}\left(\frac{R_{Es}}{D}\right)^3$$

and that this value is 2.2. R_{Es} is the distance between the Sun and Earth, and M_s is the Sun's mass.

5-19. The orbital revolution of the Moon about Earth takes about 27.3 days and is in the same direction as Earth's rotation (24 h). Use this information to show that high tides occur everywhere on Earth every 12 h and 26 min.

5-20. A thin disk of mass M and radius R lies in the (x, y) plane with the z-axis passing through the center of the disk. Calculate the gravitational potential $\Phi(z)$ and the gravitational field $\mathbf{g}(z) = -\nabla\Phi(z) = -\hat{\mathbf{k}}\, d\Phi(z)/dz$ on the z-axis.

5-21. A point mass m is located a distance D from the nearest end of a thin rod of mass M and length L along the axis of the rod. Find the gravitational force exerted on the point mass by the rod.