PROBLEMS

SECTION 3.1. Continuous Random Variables and PDFs

Problem 1. Let X be uniformly distributed in the unit interval $[0,1]$. Consider the random variable $Y = g(X)$, where

$$g(x) = \begin{cases} 1, & \text{if } x \le 1/3, \\ 2, & \text{if } x > 1/3. \end{cases}$$

Find the expected value of Y by first deriving its PMF. Verify the result using the expected value rule.

Problem 2. Laplace random variable. Let X have the PDF

$$f_X(x) = \frac{\lambda}{2} e^{-\lambda|x|},$$

where λ is a positive scalar. Verify that f_X satisfies the normalization condition, and evaluate the mean and variance of X.

Problem 3.* Show that the expected value of a discrete or continuous random variable X satisfies

$$\mathbf{E}[X] = \int_0^\infty \mathbf{P}(X > x)\,dx - \int_0^\infty \mathbf{P}(X < -x)\,dx.$$

Solution. Suppose that X is continuous. We then have

$$\begin{aligned}\int_0^\infty \mathbf{P}(X > x)\,dx &= \int_0^\infty \left(\int_x^\infty f_X(y)\,dy\right) dx \\ &= \int_0^\infty \left(\int_0^y f_X(y)\,dx\right) dy \\ &= \int_0^\infty f_X(y) \left(\int_0^y dx\right) dy \\ &= \int_0^\infty y f_X(y)\,dy,\end{aligned}$$

where for the second equality we have reversed the order of integration by writing the set $\{(x,y) \mid 0 \le x < \infty,\ x \le y < \infty\}$ as $\{(x,y) \mid 0 \le x \le y,\ 0 \le y < \infty\}$. Similarly, we can show that

$$\int_0^\infty \mathbf{P}(X < -x)\,dx = -\int_{-\infty}^0 y f_X(y)\,dy.$$

Combining the two relations above, we obtain the desired result.

If X is discrete, we have

$$\begin{aligned}\mathbf{P}(X > x) &= \int_0^\infty \sum_{y>x} p_X(y) \\ &= \sum_{y>0}\left(\int_0^y p_X(y)\,dx\right) \\ &= \sum_{y>0} p_X(y)\left(\int_0^y dx\right) \\ &= \sum_{y>0} p_X(y) y,\end{aligned}$$

and the rest of the argument is similar to the continuous case.

Problem 4.* Establish the validity of the expected value rule

$$\mathbf{E}\big[g(X)\big] = \int_{-\infty}^\infty g(x) f_X(x)\,dx,$$

where X is a continuous random variable with PDF f_X.

Solution. Let us express the function g as the difference of two nonnegative functions,

$$g(x) = g^+(x) - g^-(x),$$

where $g^+(x) = \max\{g(x), 0\}$, and $g^-(x) = \max\{-g(x), 0\}$. In particular, for any $t \geq 0$, we have $g(x) > t$ if and only if $g^+(x) > t$.

We will use the result

$$\mathbf{E}\big[g(X)\big] = \int_0^\infty \mathbf{P}\big(g(X) > t\big)\,dt - \int_0^\infty \mathbf{P}\big(g(X) < -t\big)\,dt$$

from the preceding problem. The first term in the right-hand side is equal to

$$\int_0^\infty \int_{\{x \,|\, g(x)>t\}} f_X(x)\,dx\,dt = \int_{-\infty}^\infty \int_{\{t \,|\, 0\leq t<g(x)\}} f_X(x)\,dt\,dx = \int_{-\infty}^\infty g^+(x) f_X(x)\,dx.$$

By a symmetrical argument, the second term in the right-hand side is given by

$$\int_0^\infty \mathbf{P}\big(g(X) < -t\big)\,dt = \int_{-\infty}^\infty g^-(x) f_X(x)\,dx.$$

Combining the above equalities, we obtain

$$\mathbf{E}\big[g(X)\big] = \int_{-\infty}^\infty g^+(x) f_X(x)\,dx - \int_{-\infty}^\infty g^-(x) f_X(x)\,dx = \int_{-\infty}^\infty g(x) f_X(x)\,dx.$$

SECTION 3.2. Cumulative Distribution Functions

Problem 5. Consider a triangle and a point chosen within the triangle according to the uniform probability law. Let X be the distance from the point to the base of the triangle. Given the height of the triangle, find the CDF and the PDF of X.

Problem 6. Calamity Jane goes to the bank to make a withdrawal, and is equally likely to find 0 or 1 customers ahead of her. The service time of the customer ahead, if present, is exponentially distributed with parameter λ. What is the CDF of Jane's waiting time?

Problem 7. Alvin throws darts at a circular target of radius r and is equally likely to hit any point in the target. Let X be the distance of Alvin's hit from the center.

(a) Find the PDF, the mean, and the variance of X.

(b) The target has an inner circle of radius t. If $X \le t$, Alvin gets a score of $S = 1/X$. Otherwise his score is $S = 0$. Find the CDF of S. Is S a continuous random variable?

Problem 8. Consider two continuous random variables Y and Z, and a random variable X that is equal to Y with probability p and to Z with probability $1 - p$.

(a) Show that the PDF of X is given by

$$f_X(x) = pf_Y(x) + (1-p)f_Z(x).$$

(b) Calculate the CDF of the two-sided exponential random variable that has PDF given by

$$f_X(x) = \begin{cases} p\lambda e^{\lambda x}, & \text{if } x < 0, \\ (1-p)\lambda e^{-\lambda x}, & \text{if } x \ge 0, \end{cases}$$

where $\lambda > 0$ and $0 < p < 1$.

Problem 9.* Mixed random variables. Probabilistic models sometimes involve random variables that can be viewed as a mixture of a discrete random variable Y and a continuous random variable Z. By this we mean that the value of X is obtained according to the probability law of Y with a given probability p, and according to the probability law of Z with the complementary probability $1 - p$. Then, X is called a *mixed random variable* and its CDF is given, using the total probability theorem, by

$$\begin{aligned} F_X(x) &= \mathbf{P}(X \le x) \\ &= p\mathbf{P}(Y \le x) + (1-p)\mathbf{P}(Z \le x) \\ &= pF_Y(x) + (1-p)F_Z(x). \end{aligned}$$

Its expected value is defined in a way that conforms to the total expectation theorem:

$$\mathbf{E}[X] = p\mathbf{E}[Y] + (1-p)\mathbf{E}[Z].$$

The taxi stand and the bus stop near Al's home are in the same location. Al goes there at a given time and if a taxi is waiting (this happens with probability 2/3) he boards it. Otherwise he waits for a taxi or a bus to come, whichever comes first. The next taxi will arrive in a time that is uniformly distributed between 0 and 10 minutes, while the next bus will arrive in exactly 5 minutes. Find the CDF and the expected value of Al's waiting time.

Solution. Let A be the event that Al will find a taxi waiting or will be picked up by the bus after 5 minutes. Note that the probability of boarding the next bus, given that Al has to wait, is

$$\mathbf{P}(\text{a taxi will take more than 5 minutes to arrive}) = \frac{1}{2}.$$

Al's waiting time, call it X, is a mixed random variable. With probability

$$\mathbf{P}(A) = \frac{2}{3} + \frac{1}{3} \cdot \frac{1}{2} = \frac{5}{6},$$

it is equal to its discrete component Y (corresponding to either finding a taxi waiting, or boarding the bus), which has PMF

$$p_Y(y) = \begin{cases} \dfrac{2}{3\mathbf{P}(A)}, & \text{if } y = 0, \\ \dfrac{1}{6\mathbf{P}(A)}, & \text{if } y = 5, \end{cases}$$

$$= \begin{cases} \dfrac{12}{15}, & \text{if } y = 0, \\ \dfrac{3}{15}, & \text{if } y = 5. \end{cases}$$

[This equation follows from the calculation

$$p_Y(0) = \mathbf{P}(Y = 0 \mid A) = \frac{\mathbf{P}(Y = 0, A)}{\mathbf{P}(A)} = \frac{2}{3\mathbf{P}(A)}.$$

The calculation for $p_Y(5)$ is similar.] With the complementary probability $1 - \mathbf{P}(A)$, the waiting time is equal to its continuous component Z (corresponding to boarding a taxi after having to wait for some time less than 5 minutes), which has PDF

$$f_Z(z) = \begin{cases} 1/5, & \text{if } 0 \le z \le 5, \\ 0, & \text{otherwise.} \end{cases}$$

The CDF is given by $F_X(x) = \mathbf{P}(A)F_Y(x) + \big(1 - \mathbf{P}(A)\big)F_Z(x)$, from which

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0, \\ \dfrac{5}{6} \cdot \dfrac{12}{15} + \dfrac{1}{6} \cdot \dfrac{x}{5}, & \text{if } 0 \le x < 5, \\ 1, & \text{if } 5 \le x. \end{cases}$$

The expected value of the waiting time is

$$\mathbf{E}[X] = \mathbf{P}(A)\mathbf{E}[Y] + \big(1 - \mathbf{P}(A)\big)\mathbf{E}[Z] = \frac{5}{6} \cdot \frac{3}{15} \cdot 5 + \frac{1}{6} \cdot \frac{5}{2} = \frac{15}{12}.$$

Problem 10.* Simulating a continuous random variable. A computer has a subroutine that can generate values of a random variable U that is uniformly distributed in the interval $[0, 1]$. Such a subroutine can be used to generate values of a continuous random variable with given CDF $F(x)$ as follows. If U takes a value u, we let the value of X be a number x that satisfies $F(x) = u$. For simplicity, we assume that the given CDF is strictly increasing over the range S of values of interest, where $S = \{x \,|\, 0 < F(x) < 1\}$. This condition guarantees that for any $u \in (0, 1)$, there is a unique x that satisfies $F(x) = u$.

(a) Show that the CDF of the random variable X thus generated is indeed equal to the given CDF.

(b) Describe how this procedure can be used to simulate an exponential random variable with parameter λ.

(c) How can this procedure be generalized to simulate a discrete integer-valued random variable?

Solution. (a) By definition, the random variables X and U satisfy the relation $F(X) = U$. Since F is strictly increasing, we have for every x,

$$X \leq x \qquad \text{if and only if} \qquad F(X) \leq F(x).$$

Therefore,

$$\mathbf{P}(X \leq x) = \mathbf{P}\big(F(X) \leq F(x)\big) = \mathbf{P}\big(U \leq F(x)\big) = F(x),$$

where the last equality follows because U is uniform. Thus, X has the desired CDF.

(b) The exponential CDF has the form $F(x) = 1 - e^{-\lambda x}$ for $x \geq 0$. Thus, to generate values of X, we should generate values $u \in (0, 1)$ of a uniformly distributed random variable U, and set X to the value for which $1 - e^{-\lambda x} = u$, or $x = -\ln(1 - u)/\lambda$.

(c) Let again F be the desired CDF. To any $u \in (0, 1)$, there corresponds a unique integer x_u such that $F(x_u - 1) < u \leq F(x_u)$. This correspondence defines a random variable X as a function of the random variable U. We then have, for every integer k,

$$\mathbf{P}(X = k) = \mathbf{P}\big(F(k-1) < U \leq F(k)\big) = F(k) - F(k-1).$$

Therefore, the CDF of X is equal to F, as desired.

SECTION 3.3. Normal Random Variables

Problem 11. Let X and Y be normal random variables with means 0 and 1, respectively, and variances 1 and 4, respectively.

(a) Find $\mathbf{P}(X \leq 1.5)$ and $\mathbf{P}(X \leq -1)$.

(b) Find the PDF of $(Y - 1)/2$.

(c) Find $\mathbf{P}(-1 \leq Y \leq 1)$.

Problem 12. Let X be a normal random variable with zero mean and standard deviation σ. Use the normal tables to compute the probabilities of the events $\{X \geq k\sigma\}$ and $\{|X| \leq k\sigma\}$ for $k = 1, 2, 3$.

Problem 13. A city's temperature is modeled as a normal random variable with mean and standard deviation both equal to 10 degrees Celsius. What is the probability that the temperature at a randomly chosen time will be less than or equal to 59 degrees Fahrenheit?

Problem 14.* Show that the normal PDF satisfies the normalization property. *Hint:* The integral $\int_{-\infty}^{\infty} e^{-x^2/2}\,dx$ is equal to the square root of

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2/2}e^{-y^2/2}\,dx\,dy,$$

and the latter integral can be evaluated by transforming to polar coordinates.

Solution. We note that

$$\begin{aligned}\left(\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,dx\right)^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-x^2/2}\,dx\int_{-\infty}^{\infty}e^{-y^2/2}\,dy\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-(x^2+y^2)/2}\,dx\,dy\\ &= \frac{1}{2\pi}\int_{0}^{2\pi}\int_{0}^{\infty}e^{-r^2/2}r\,dr\,d\theta\\ &= \int_{0}^{\infty}e^{-r^2/2}\,r\,dr\\ &= \int_{0}^{\infty}e^{-u}\,du\\ &= -e^{-u}\Big|_0^{\infty}\\ &= 1,\end{aligned}$$

where for the third equality, we use a transformation into polar coordinates, and for the fifth equality, we use the change of variables $u = r^2/2$. Thus, we have

$$\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,dx = 1,$$

because the integral is positive. Using the change of variables $u = (x-\mu)/\sigma$, it follows that

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\,\sigma}e^{-(x-\mu)^2/2\sigma^2}\,dx = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-u^2/2}du = 1.$$

SECTION 3.4. Joint PDFs of Multiple Random Variables

Problem 15. A point is chosen at random (according to a uniform PDF) within a semicircle of the form $\{(x,y)\mid x^2+y^2\le r^2,\ y\ge 0\}$, for some given $r>0$.

(a) Find the joint PDF of the coordinates X and Y of the chosen point.

(b) Find the marginal PDF of Y and use it to find $\mathbf{E}[Y]$.

(c) Check your answer in (b) by computing $\mathbf{E}[Y]$ directly without using the marginal PDF of Y.

Problem 16. Consider the following variant of Buffon's needle problem (Example 3.11), which was investigated by Laplace. A needle of length l is dropped on a plane surface that is partitioned in rectangles by horizontal lines that are a apart and vertical lines that are b apart. Suppose that the needle's length l satisfies $l < a$ and $l < b$. What is the expected number of rectangle sides crossed by the needle? What is the probability that the needle will cross at least one side of some rectangle?

Problem 17.* Estimating an expected value by simulation using samples of another random variable. Let $Y_1, \ldots, Y_n$ be independent random variables drawn from a common and known PDF f_Y. Let S be the set of all possible values of Y_i, $S = \{y \,|\, f_Y(y) > 0\}$. Let X be a random variable with known PDF f_X, such that $f_X(y) = 0$, for all $y \notin S$. Consider the random variable

$$Z = \frac{1}{n}\sum_{i=1}^{n} Y_i \frac{f_X(Y_i)}{f_Y(Y_i)}.$$

Show that

$$\mathbf{E}[Z] = \mathbf{E}[X].$$

Solution. We have

$$\mathbf{E}\left[Y_i \frac{f_X(Y_i)}{f_Y(Y_i)}\right] = \int_S y \frac{f_X(y)}{f_Y(y)} f_Y(y)\,dy = \int_S y f_X(y)\,dy = \mathbf{E}[X].$$

Thus,

$$\mathbf{E}[Z] = \frac{1}{n}\sum_{i=1}^{n} \mathbf{E}\left[Y_i \frac{f_X(Y_i)}{f_Y(Y_i)}\right] = \frac{1}{n}\sum_{i=1}^{n} \mathbf{E}[X] = \mathbf{E}[X].$$

SECTION 3.5. Conditioning

Problem 18. Let X be a random variable with PDF

$$f_X(x) = \begin{cases} x/4, & \text{if } 1 < x \le 3, \\ 0, & \text{otherwise,} \end{cases}$$

and let A be the event $\{X \ge 2\}$.

(a) Find $\mathbf{E}[X]$, $\mathbf{P}(A)$, $f_{X|A}(x)$, and $\mathbf{E}[X \,|\, A]$.

(b) Let $Y = X^2$. Find $\mathbf{E}[Y]$ and $\mathrm{var}(Y)$.

Problem 19. The random variable X has the PDF

$$f_X(x) = \begin{cases} cx^{-2}, & \text{if } 1 \le x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Determine the value of c.

(b) Let A be the event $\{X > 1.5\}$. Calculate $\mathbf{P}(A)$ and the conditional PDF of X given that A has occurred.

(c) Let $Y = X^2$. Calculate the conditional expectation and the conditional variance of Y given A.

Problem 20. An absent-minded professor schedules two student appointments for the same time. The appointment durations are independent and exponentially distributed with mean thirty minutes. The first student arrives on time, but the second student arrives five minutes late. What is the expected value of the time between the arrival of the first student and the departure of the second student?

Problem 21. We start with a stick of length ℓ. We break it at a point which is chosen according to a uniform distribution and keep the piece, of length Y, that contains the left end of the stick. We then repeat the same process on the piece that we were left with, and let X be the length of the remaining piece after breaking for the second time.

(a) Find the joint PDF of Y and X.

(b) Find the marginal PDF of X.

(c) Use the PDF of X to evaluate $\mathbf{E}[X]$.

(d) Evaluate $\mathbf{E}[X]$, by exploiting the relation $X = Y \cdot (X/Y)$.

Problem 22. We have a stick of unit length, and we consider breaking it in three pieces using one of the following three methods.

(i) We choose randomly and independently two points on the stick using a uniform PDF, and we break the stick at these two points.

(ii) We break the stick at a random point chosen by using a uniform PDF, and then we break the piece that contains the right end of the stick. at a random point chosen by using a uniform PDF.

(iii) We break the stick at a random point chosen by using a uniform PDF, and then we break the larger of the two pieces at a random point chosen by using a uniform PDF.

For each of the methods (i), (ii), and (iii), what is the probability that the three pieces we are left with can form a triangle?

Problem 23. Let the random variables X and Y have a joint PDF which is uniform over the triangle with vertices at $(0,0)$, $(0,1)$, and (1.0).

(a) Find the joint PDF of X and Y.

(b) Find the marginal PDF of Y.

(c) Find the conditional PDF of X given Y.

(d) Find $\mathbf{E}[X \mid Y = y]$, and use the total expectation theorem to find $\mathbf{E}[X]$ in terms of $\mathbf{E}[Y]$.

(e) Use the symmetry of the problem to find the value of $\mathbf{E}[X]$.

Problem 24. Let X and Y be two random variables that are uniformly distributed over the triangle formed by the points $(0,0)$, $(1,0)$, and $(0,2)$ (this is an asymmetric version of the PDF in the previous problem). Calculate $\mathbf{E}[X]$ and $\mathbf{E}[Y]$ by following the same steps as in the previous problem.

Problem 25. The coordinates X and Y of a point are independent zero mean normal random variables with common variance σ^2. Given that the point is at a distance of at least c from the origin, find the conditional joint PDF of X and Y.

Problem 26.* Let $X_1,\ldots,X_n$ be independent random variables. Show that

$$\frac{\mathrm{var}\left(\prod_{i=1}^n X_i\right)}{\prod_{i=1}^n \mathbf{E}[X_i^2]} = \prod_{i=1}^n \left(\frac{\mathrm{var}(X_i)}{\mathbf{E}[X_i^2]} + 1\right) - 1.$$

Solution. We have

$$\begin{aligned}
\mathrm{var}\left(\prod_{i=1}^n X_i\right) &= \mathbf{E}\left[\prod_{i=1}^n X_i^2\right] - \prod_{i=1}^n \left(\mathbf{E}[X_i]\right)^2 \\
&= \prod_{i=1}^n \mathbf{E}[X_i^2] - \prod_{i=1}^n \left(\mathbf{E}[X_i]\right)^2 \\
&= \prod_{i=1}^n \left(\mathrm{var}(X_i) + \left(\mathbf{E}[X_i]\right)^2\right) - \prod_{i=1}^n \left(\mathbf{E}[X_i]\right)^2.
\end{aligned}$$

The desired result follows by dividing both sides by

$$\prod_{i=1}^n \left(\mathbf{E}[X_i]\right)^2.$$

Problem 27.* **Conditioning multiple random variables on events.** Let X and Y be continuous random variables with joint PDF $f_{X,Y}$, let A be a subset of the two-dimensional plane, and let $C = \{(X,Y) \in A\}$. Assume that $\mathbf{P}(C) > 0$, and define

$$f_{X,Y|C}(x,y) = \begin{cases} \dfrac{f_{X,Y}(x,y)}{\mathbf{P}(C)}, & \text{if } (x,y) \in A, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Show that $f_{X,Y|C}$ is a legitimate joint PDF.

(b) Consider a partition of the two-dimensional plane into disjoint subsets A_i, $i = 1,\ldots,n$, let $C_i = \{(X,Y) \in A_i\}$, and assume that $\mathbf{P}(C_i) > 0$ for all i. Derive the following version of the total probability theorem

$$f_{X,Y}(x,y) = \sum_{i=1}^n \mathbf{P}(C_i) f_{X,Y|C_i}(x,y).$$

Problem 28.* Consider the following two-sided exponential PDF

$$f_X(x) = \begin{cases} p\lambda e^{-\lambda x}, & \text{if } x \geq 0, \\ (1-p)\lambda e^{\lambda x}, & \text{if } x < 0, \end{cases}$$

where λ and p are scalars with $\lambda > 0$ and $p \in [0, 1]$. Find the mean and the variance of X in two ways:

(a) By straightforward calculation of the associated expected values.

(b) By using a divide-and-conquer strategy, and the mean and variance of the (one-sided) exponential random variable.

Solution. (a)

$$\begin{aligned} \mathbf{E}[X] &= \int_{-\infty}^{\infty} x f_X(x)\,dx \\ &= \int_{-\infty}^{0} x(1-p)\lambda e^{\lambda x}\,dx + \int_{0}^{\infty} xp\lambda e^{-\lambda x}\,dx \\ &= -\frac{1-p}{\lambda} + \frac{p}{\lambda} \\ &= \frac{2p-1}{\lambda}. \end{aligned}$$

$$\begin{aligned} \mathbf{E}[X^2] &= \int_{-\infty}^{\infty} x^2 f_X(x)\,dx \\ &= \int_{-\infty}^{0} x^2(1-p)\lambda e^{\lambda x}\,dx + \int_{0}^{\infty} x^2 p\lambda e^{-\lambda x}\,dx \\ &= \frac{2(1-p)}{\lambda^2} + \frac{2p}{\lambda^2} \\ &= \frac{2}{\lambda^2}, \end{aligned}$$

and

$$\text{var}(X) = \frac{2}{\lambda^2} - \left(\frac{2p-1}{\lambda}\right)^2.$$

(b) Let A be the event $\{X \geq 0\}$, and note that $\mathbf{P}(A) = p$. Conditioned on A, the random variable X has a (one-sided) exponential distribution with parameter λ. Also, conditioned on A^c, the random variable $-X$ has the same one-sided exponential distribution. Thus,

$$\mathbf{E}[X \mid A] = \frac{1}{\lambda}, \qquad \mathbf{E}[X \mid A^c] = -\frac{1}{\lambda},$$

and

$$\mathbf{E}[X^2 \mid A] = \mathbf{E}[X^2 \mid A^c] = \frac{2}{\lambda^2}.$$

It follows that

$$\begin{aligned} \mathbf{E}[X] &= \mathbf{P}(A)\mathbf{E}[X \mid A] + \mathbf{P}(A^c)\mathbf{E}[X \mid A^c] \\ &= \frac{p}{\lambda} - \frac{1-p}{\lambda} \\ &= \frac{2p-1}{\lambda}, \end{aligned}$$

$$\begin{aligned}\mathbf{E}[X^2] &= \mathbf{P}(A)\mathbf{E}[X^2 \mid A] + \mathbf{P}(A^c)\mathbf{E}[X^2 \mid A^c] \\ &= \frac{2p}{\lambda^2} + \frac{2(1-p)}{\lambda^2} \\ &= \frac{2}{\lambda^2},\end{aligned}$$

and

$$\text{var}(X) = \frac{2}{\lambda^2} - \left(\frac{2p-1}{\lambda}\right)^2.$$

Problem 29.* Let X, Y, and Z be three random variables with joint PDF $f_{X,Y,Z}$. Show the multiplication rule:

$$f_{X,Y,Z}(x, y, z) = f_{X|Y,Z}(x \mid y, z) f_{Y|Z}(y \mid z) f_Z(z).$$

Solution. We have, using the definition of conditional density,

$$f_{X|Y,Z}(x \mid y, z) = \frac{f_{X,Y,Z}(x, y, z)}{f_{Y,Z}(y, z)},$$

and

$$f_{Y,Z}(y, z) = f_{Y|Z}(y \mid z) f_Z(z).$$

Combining these two relations, we obtain the multiplication rule.

Problem 30.* **The Beta PDF.** The beta PDF with parameters $\alpha > 0$ and $\beta > 0$ has the form

$$f_X(x) = \begin{cases} \dfrac{1}{\mathrm{B}(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1}, & \text{if } 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

The normalizing constant is

$$\mathrm{B}(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\, dx,$$

and is known as the Beta function.

(a) Show that for any $m > 0$, the mth moment of X is given by

$$\mathbf{E}[X^m] = \frac{\mathrm{B}(\alpha + m, \beta)}{\mathrm{B}(\alpha, \beta)}.$$

(b) Assume that α and β are integer. Show that

$$\mathrm{B}(\alpha, \beta) = \frac{(\alpha-1)!\,(\beta-1)!}{(\alpha+\beta-1)!},$$

so that

$$\mathbf{E}[X^m] = \frac{\alpha(\alpha+1)\cdots(\alpha+m-1)}{(\alpha+\beta)(\alpha+\beta+1)\cdots(\alpha+\beta+m-1)}.$$

(Recall here the convention that $0! = 1$.)

Solution. (a) We have

$$\mathbf{E}[X^m] = \frac{1}{\mathrm{B}(\alpha,\beta)} \int_0^1 x^m x^{\alpha-1}(1-x)^{\beta-1}\,dx = \frac{\mathrm{B}(\alpha+m,\beta)}{\mathrm{B}(\alpha,\beta)}.$$

(b) In the special case where $\alpha = 1$ or $\beta = 1$, we can carry out the straightforward integration in the definition of $\mathrm{B}(\alpha,\beta)$, and verify the result. We will now deal with the general case. Let $Y, Y_1, \ldots, Y_{\alpha+\beta}$ be independent random variables, uniformly distributed over the interval $[0,1]$, and let A be the event

$$A = \{Y_1 \leq \cdots \leq Y_\alpha \leq Y \leq Y_{\alpha+1} \leq \cdots \leq Y_{\alpha+\beta}\}.$$

Then,

$$\mathbf{P}(A) = \frac{1}{(\alpha+\beta+1)!},$$

because all ways of ordering these $\alpha+\beta+1$ random variables are equally likely.

Consider the following two events:

$$B = \Big\{ \max\{Y_1,\ldots,Y_\alpha\} \leq Y \Big\}, \qquad C = \Big\{ Y \leq \min\{Y_{\alpha+1},\ldots,Y_{\alpha+\beta}\} \Big\}.$$

We have, using the total probability theorem,

$$\begin{aligned}
\mathbf{P}(B\cap C) &= \int_0^1 \mathbf{P}(B\cap C \,|\, Y=y) f_Y(y)\,dy \\
&= \int_0^1 \mathbf{P}\big(\max\{Y_1,\ldots,Y_\alpha\} \leq y \leq \min\{Y_{\alpha+1},\ldots,Y_{\alpha+\beta}\}\big)\,dy \\
&= \int_0^1 \mathbf{P}\big(\max\{Y_1,\ldots,Y_\alpha\} \leq y\big)\,\mathbf{P}\big(y \leq \min\{Y_{\alpha+1},\ldots,Y_{\alpha+\beta}\}\big)\,dy \\
&= \int_0^1 y^\alpha (1-y)^\beta\,dy.
\end{aligned}$$

We also have

$$\mathbf{P}(A \,|\, B\cap C) = \frac{1}{\alpha!\,\beta!},$$

because given the events B and C, all $\alpha!$ possible orderings of $Y_1,\ldots,Y_\alpha$ are equally likely, and all $\beta!$ possible orderings of $Y_{\alpha+1},\ldots,Y_{\alpha+\beta}$ are equally likely.

By writing the equation

$$\mathbf{P}(A) = \mathbf{P}(B\cap C)\,\mathbf{P}(A \,|\, B\cap C)$$

in terms of the preceding relations, we finally obtain

$$\frac{1}{(\alpha+\beta+1)!} = \frac{1}{\alpha!\,\beta!} \int_0^1 y^\alpha (1-y)^\beta\,dy,$$

or

$$\int_0^1 y^\alpha (1-y)^\beta \, dy = \frac{\alpha!\,\beta!}{(\alpha+\beta+1)!}.$$

This equation can be written as

$$B(\alpha+1, \beta+1) = \frac{\alpha!\,\beta!}{(\alpha+\beta+1)!}, \qquad \text{for all integer } \alpha > 0,\ \beta > 0.$$

Problem 31.* Estimating an expected value by simulation. Let $f_X(x)$ be a PDF such that for some nonnegative scalars a, b, and c, we have $f_X(x) = 0$ for all x outside the interval $[a, b]$, and $xf_X(x) \le c$ for all x. Let Y_i, $i = 1, \ldots, n$, be independent random variables with values generated as follows: a point (V_i, W_i) is chosen at random (according to a uniform PDF) within the rectangle whose corners are $(a, 0)$, $(b, 0)$, (a, c), and (b, c), and if $W_i \le V_i f_X(V_i)$, the value of Y_i is set to 1, and otherwise it is set to 0. Consider the random variable

$$Z = \frac{Y_1 + \cdots + Y_n}{n}.$$

Show that

$$\mathbf{E}[Z] = \frac{\mathbf{E}[X]}{c(b-a)}$$

and

$$\text{var}(Z) \le \frac{1}{4n}.$$

In particular, we have $\text{var}(Z) \to 0$ as $n \to \infty$.

Solution. We have

$$\begin{aligned}
\mathbf{P}(Y_i = 1) &= \mathbf{P}\big(W_i \le V_i f_X(V_i)\big) \\
&= \int_a^b \int_0^{v f_X(v)} \frac{1}{c(b-a)} \, dw \, dv \\
&= \frac{\int_a^b v f_X(v)\, dv}{c(b-a)} \\
&= \frac{\mathbf{E}[X]}{c(b-a)}.
\end{aligned}$$

The random variable Z has mean $\mathbf{P}(Y_i = 1)$ and variance

$$\text{var}(Z) = \frac{\mathbf{P}(Y_i = 1)\big(1 - \mathbf{P}(Y_i = 1)\big)}{n}.$$

Since $0 \le (1-2p)^2 = 1 - 4p(1-p)$, we have $p(1-p) \le 1/4$ for any p in $[0, 1]$, so it follows that $\text{var}(Z) \le 1/(4n)$.

Problem 32.* Let X and Y be continuous random variables with joint PDF $f_{X,Y}$. Suppose that for any subsets A and B of the real line, the events $\{X \in A\}$ and $\{Y \in B\}$ are independent. Show that the random variables X and Y are independent.

Solution. For any two real numbers x and y, using the independence of the events $\{X \leq x\}$ and $\{Y \leq y\}$, we have

$$F_{X,Y}(x,y) = \mathbf{P}(X \leq x, Y \leq y) = \mathbf{P}(X \leq x)\,\mathbf{P}(Y \leq y) = F_X(x)F_Y(y).$$

Taking derivatives of both sides, we obtain

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}}{\partial x \partial y}(x,y) = \frac{\partial F_X}{\partial x}(x)\frac{\partial F_Y}{\partial y}(y) = f_X(x)f_Y(y),$$

which establishes that X and Y are independent.

Problem 33.* The sum of a random number of random variables. You visit a random number N of stores and in the ith store, you spend a random amount of money X_i. Let

$$T = X_1 + X_2 + \cdots + X_N$$

be the total amount of money that you spend. We assume that N is a positive integer random variable with a given PMF, and that the X_i are random variables with the same mean $\mathbf{E}[X]$ and variance $\text{var}(X)$. Furthermore, we assume that N and all the X_i are independent. Show that

$$\mathbf{E}[T] = \mathbf{E}[X]\,\mathbf{E}[N], \qquad \text{and} \qquad \text{var}(T) = \text{var}(X)\,\mathbf{E}[N] + \big(\mathbf{E}[X]\big)^2\text{var}(N).$$

Solution. We have for all i,

$$\mathbf{E}[T \mid N = i] = i\mathbf{E}[X],$$

since conditional on $N = i$, you will visit exactly i stores, and you will spend an expected amount of money $\mathbf{E}[X]$ in each.

We now apply the total expectation theorem. We have

$$\begin{aligned}
\mathbf{E}[T] &= \sum_{i=1}^{\infty} \mathbf{P}(N = i)\,\mathbf{E}[T \mid N = i] \\
&= \sum_{i=1}^{\infty} \mathbf{P}(N = i) i\mathbf{E}[X] \\
&= \mathbf{E}[X] \sum_{i=1}^{\infty} i\mathbf{P}(N = i) \\
&= \mathbf{E}[X]\,\mathbf{E}[N].
\end{aligned}$$

Similarly, using also the independence of the X_i, which implies that $\mathbf{E}[X_iX_j] = \big(\mathbf{E}[X]\big)^2$ if $i \neq j$, the second moment of T is calculated as

$$\begin{aligned}
\mathbf{E}[T^2] &= \sum_{i=1}^{\infty} \mathbf{P}(N=i)\,\mathbf{E}[T^2 \mid N=i] \\
&= \sum_{i=1}^{\infty} \mathbf{P}(N=i)\,\mathbf{E}\big[(X_1+\cdots+X_N)^2 \mid N=i\big] \\
&= \sum_{i=1}^{\infty} \mathbf{P}(N=i)\big(i\mathbf{E}[X^2] + i(i-1)\big(\mathbf{E}[X]\big)^2\big) \\
&= \mathbf{E}[X^2]\sum_{i=1}^{\infty} i\mathbf{P}(N=i) + \big(\mathbf{E}[X]\big)^2 \sum_{i=1}^{\infty} i(i-1)\mathbf{P}(N=i) \\
&= \mathbf{E}[X^2]\,\mathbf{E}[N] + \big(\mathbf{E}[X]\big)^2\big(\mathbf{E}[N^2]-\mathbf{E}[N]\big) \\
&= \mathrm{var}(X)\,\mathbf{E}[N] + \big(\mathbf{E}[X]\big)^2\mathbf{E}[N^2].
\end{aligned}$$

The variance is then obtained by

$$\begin{aligned}
\mathrm{var}(T) &= \mathbf{E}[T^2] - \big(\mathbf{E}[T]\big)^2 \\
&= \mathrm{var}(X)\,\mathbf{E}[N] + \big(\mathbf{E}[X]\big)^2\mathbf{E}[N^2] - \big(\mathbf{E}[X]\big)^2\big(\mathbf{E}[N]\big)^2 \\
&= \mathrm{var}(X)\,\mathbf{E}[N] + \big(\mathbf{E}[X]\big)^2\Big(\mathbf{E}[N^2] - \big(\mathbf{E}[N]\big)^2\Big),
\end{aligned}$$

so finally

$$\mathrm{var}(T) = \mathrm{var}(X)\,\mathbf{E}[N] + \big(\mathbf{E}[X]\big)^2\mathrm{var}(N).$$

Note: The formulas for $\mathbf{E}[T]$ and $\mathrm{var}(T)$ will also be obtained in Chapter 4, using a more abstract approach.

SECTION 3.6. The Continuous Bayes' Rule

Problem 34. A defective coin minting machine produces coins whose probability of heads is a random variable P with PDF

$$f_P(p) = \begin{cases} pe^p, & p \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

A coin produced by this machine is selected and tossed repeatedly, with successive tosses assumed independent.

(a) Find the probability that a coin toss results in heads.

(b) Given that a coin toss resulted in heads, find the conditional PDF of P.

(c) Given that the first coin toss resulted in heads, find the conditional probability of heads on the next toss.

Problem 35.* Let X and Y be independent continuous random variables with PDFs f_X and f_Y, respectively, and let $Z = X + Y$.

(a) Show that $f_{Z|X}(z\,|\,x) = f_Y(z-x)$. *Hint:* Write an expression for the conditional CDF of Z given X, and differentiate.

(b) Assume that X and Y are exponentially distributed with parameter λ. Find the conditional PDF of X, given that $Z = z$.

(c) Assume that X and Y are normal random variables with mean zero and variances σ_x^2 and σ_y^2, respectively. Find the conditional PDF of X, given that $Z = z$.

Solution. (a) We have

$$\begin{aligned}\mathbf{P}(Z \le z\,|\,X = x) &= \mathbf{P}(X + Y \le z\,|\,X = x)\\ &= \mathbf{P}(x + Y \le z\,|\,X = x)\\ &= \mathbf{P}(x + Y \le z)\\ &= \mathbf{P}(Y \le z - x),\end{aligned}$$

where the third equality follows from the independence of X and Y. By differentiating both sides with respect to z, the result follows.

(b) We have, for $0 \le x \le z$,

$$f_{X|Z}(x\,|\,z) = \frac{f_{Z|X}(z\,|\,x)f_X(x)}{f_Z(z)} = \frac{f_Y(z-x)f_X(x)}{f_Z(z)} = \frac{\lambda e^{-\lambda(z-x)}\lambda e^{-\lambda x}}{f_Z(z)} = \frac{\lambda^2 e^{-\lambda z}}{f_Z(z)}.$$

Since this is the same for all x, it follows that the conditional distribution of X is uniform on the interval $[0, z]$, with PDF $f_{X|Z}(x\,|\,z) = 1/z$.

(c) We have

$$f_{X|Z}(x\,|\,z) = \frac{f_Y(z-x)f_X(x)}{f_Z(z)} = \frac{1}{f_Z(z)} \cdot \frac{1}{\sqrt{2\pi}\,\sigma_y} e^{-(z-x)^2/2\sigma_y^2} \frac{1}{\sqrt{2\pi}\,\sigma_x} e^{-x^2/2\sigma_x^2}.$$

We focus on the terms in the exponent. By completing the square, we find that the negative of the exponent is of the form

$$\frac{(z-x)^2}{2\sigma_y^2} + \frac{x^2}{2\sigma_x^2} = \frac{\sigma_x^2 + \sigma_y^2}{2\sigma_x^2\sigma_y^2}\left(x - \frac{z\sigma_x^2}{\sigma_x^2 + \sigma_y^2}\right)^2 + \frac{z^2}{2\sigma_y^2}\left(1 - \frac{\sigma_x^2}{\sigma_x^2 + \sigma_y^2}\right).$$

Thus, the conditional PDF of X is of the form

$$f_{X|Z}(x\,|\,z) = c(z) \cdot \exp\left\{-\frac{\sigma_x^2 + \sigma_y^2}{2\sigma_x^2\sigma_y^2}\left(x - \frac{z\sigma_x^2}{\sigma_x^2 + \sigma_y^2}\right)^2\right\},$$

where $c(z)$ does not depend on x and plays the role of a normalizing constant. We recognize this as a normal distribution with mean

$$\mathbf{E}[X\,|\,Z = z] = \frac{\sigma_x^2}{\sigma_x^2 + \sigma_y^2}\,z,$$

and variance

$$\text{var}(X\,|\,Z = z) = \frac{\sigma_x^2\sigma_y^2}{\sigma_x^2 + \sigma_y^2}.$$